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Tomita Representation of the Arnol'd Cat Map

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The Tomita Hilbert-space representation of the Arnol'd cat map model of Benatti *et al.* is described and the operators representing physical quantities are defined for the classical and quantum cases. It is seen that the exponential decay of correlations is preserved upon quantization.

1. INTRODUCTION

A task which has initiated a considerable amount of research in the last decade has been to characterize the dynamics of quantum systems that correspond to classical systems with chaotic properties (Cerdeira *et al.*, 1991; Cvitanovi'c *et al.*, 1992; and references therein). Among the various approaches to this field, one is to investigate the explicit time evolution of a quantum system when the evolution equation of the classical correspondent is sufficiently simple to be carried over to the quantum domain. A natural starting point is to try to quantize the simplest dynamical maps known to exhibit chaotic properties. The prototype of such maps on the phase space is the Arnol'd cat map, which for its simplicity combined with its strong stochastic properties has been considered by many authors (Arnol'd and Avez, 1968; Sagdeev *et al.*, 1988; Casati *et al.*, 1979; Berry *et al.*, 1979; Ford *et al.*, 1990; Benatti *et al.*, 1991).

A systematic and mathematically elaborated method for quantization of the Arnol'd cat map has been presented by Benatti *et al.* (1991a,b). They conclude that, although the map is mixing, it is not a K-system (in the algebraic sense) for irrational values of h. Thus the classical case $h = 0$, which is known to be a K-system, is actually one of the exceptional measure-zero cases of rational h's.

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The phase space representation is the main tool for studying dynamical phenomena in classical mechanics. When we turn to quantum mechanics to see what the dynamics looks like when Planck's constant no longer is assumed to be zero, one would like to retain as much as possible of the structure of the original phase space description. This can be done by using Tomita representations (Grelland, 1992). Benatti *et al.* (1991a,b) actually use a Tomita representation, but the authors chose to work at an abstract algebraic level, without dealing with details of the representation. As a consequence, the physical content of their model is by no means obvious, in particular to readers unfamiliar with the theory of W^*-algebras. Even for the advanced reader it will be of interest to explore in more detail the possible physical content and implications of this model.

The present study is based on the formulation of the general theory of Tomita representations of classical and quantum mechanics given by Grelland (1992). We find that by using a simple bra/ket formulation instead of the mathematically rigorous C^*-algebraic approach, the theory becomes much more transparent and easier to interpret.

The phase space of the Arnol'd cat map model is the two-torus. Thus, we have to deal with periodic boundary conditions, which have led to considerable difficulties in the traditional approaches to the quantization of this system. It will be seen that special care has to be taken with respect to domain questions. Except for this, the formulation is formally similar to the Tomita representation of the $\mathbf{R}^2$ phase space. While Benatti *et al.* (1991a,b) confined themselves to identifying a unitary group generating the relevant algebra and to studying the Heisenberg picture dynamics on this algebra, we try to identify operators corresponding to physical quantities of the theory, thus facilitating the interpretation of the state functions and the introduction of the Schrödinger picture dynamics.

A question of central importance in investigating this model is: which stochastic properties are lost and which are kept upon quantization? We show that for the quantum states the property of exponential decay of correlations is kept also for the quantum system.

2. THE TOMITA KET SPACE OF THE TORUS

The Tomita ket space of the torus $T^2 = [0, 1\rangle^2$, equipped with addition and multiplication modulo 1, can be characterized by the basis

$$B = \{ |\mathbf{q}] | \mathbf{q} \in [0, 1]^2 \} \tag{2.1}$$

This basis gives rise to the representation $f(\mathbf{q}) = [\mathbf{q} | f]$. We write the two components of the vector $\mathbf{q} = (q, p)$. We also introduce the countable Fourier basis:

$$B_F = \{|\mathbf{k}]|\mathbf{k} \in \mathbf{Z}^2, [\mathbf{q}|\mathbf{k}] = \exp(2\pi i\mathbf{k}\cdot\mathbf{q})\} \tag{2.2}$$

The components of $\mathbf{k}$ will be written $\mathbf{k} = (k, l)$. The ket space and its operators are used as means for modeling physical systems which have the torus as a classical phase space. To specify such a model in detail, we select an appropriate set of operators on the ket space. We need unitary operators for representing symmetry transformations, and self-adjoint operators for representing physical quantities. When working on the torus, one has to be more careful with respect to the domain of definition for unbounded operators than is customary in ordinary quantum mechanics.

The basic self-adjoint operators which we will need are

$$\begin{aligned} Q_0|\mathbf{q}] &= q|\mathbf{q}] \\ P_0|\mathbf{q}] &= p|\mathbf{q}] \end{aligned} \tag{2.3}$$

$$\begin{aligned} D|\mathbf{k}] &= 2\pi k|\mathbf{k}] \\ E|\mathbf{k}] &= -2\pi l|\mathbf{k}] \end{aligned} \tag{2.4}$$

The operators D, E may also be written as derivation operators if they are properly restricted to the domain consisting of functions of the $\mathbf{q}$-representation that are periodic in the corresponding coordinate. The Fourier basis functions have this property. We have

$$\begin{aligned} [\mathbf{q}|D &= -i\, d/dq[\mathbf{q}| \\ [\mathbf{q}|E &= i\, d/dp[\mathbf{q}| \end{aligned} \tag{2.5}$$

The operators defined above will be written, pairwise, as vectors: $\mathbf{Q}_0 = (Q_0, P_0)$, $\mathbf{E} = (E, D)$, $\mathbf{D} = (D, -E)$. Thus, (2.4) can be written in a vector notation

$$\mathbf{Q}_0|\mathbf{q}] = |\mathbf{q}] \tag{2.6}$$

$$\mathbf{D}|\mathbf{k}] = 2\pi\mathbf{k}|\mathbf{k}] \tag{2.7}$$

The unitary operator representing a translation b and a boost mv modulo 1, $\mathbf{b} = (b, mv)$, is

$$V(\mathbf{b}) = \exp(2\pi i\mathbf{E}\cdot\mathbf{b}) \tag{2.8}$$

The conjugation operator, characteristic for a Tomita representation, can be described in the bases defined above:

$$[\mathbf{q}|J|f] = [\mathbf{q}|f]^* \tag{2.9}$$

$$[\mathbf{k}|J|f] = [-\mathbf{k}|f]^* \tag{2.10}$$

The states of the classical toral system are represented by normalized kets $|f]$ in the cone $\mathcal{P}$:

$$\mathcal{P} = \{ |f] \,|\, [\mathbf{q}|f] \geq 0 \} \tag{2.11}$$

with the expectation value

$$\langle \mathbf{Q}_0 \rangle = [f|\mathbf{Q}_0|f] \tag{2.12}$$

for a system in a state $|f]$. Of particular significance is the state $|\mathbf{0}] \in B_{\mathrm{F}}$ (i.e., $\mathbf{k} = \mathbf{0}$), of uniform phase space distribution:

$$[\mathbf{q}|\mathbf{0}] = 1 \tag{2.13}$$

We will later need the projection operator P onto the uniform state: $P = |\mathbf{0}][\mathbf{0}|$.

We are now in a position to write down a pair of quantum operators corresponding to the classical operators $\mathbf{Q}_0$:

$$\mathbf{Q} = \mathbf{Q}_0 + (h/4\pi)\mathbf{E} \tag{2.14}$$

We would like to write down the commutator $[Q, P]$, but we have to be careful about the domain. If $|f]$ is such that $Q|f]$ is in the domain of P and $P|f]$ is in the domain of Q, then

$$[Q, P]|f] = ih/2\pi|f] \tag{2.15}$$

We find it reasonable to interpret Q, P as the position and momentum operators of the quantum model.

A Fourier state does not fulfill the conditions mentioned. However, the inner product between $[Q, P]|\mathbf{k}]$ and $|\mathbf{k}]$ exists and has, perhaps surprisingly, the property

$$[\mathbf{k}|[Q, P]|\mathbf{k}] = 0 \tag{2.16}$$

Since all physically relevant operators of the quantum system are generated by the pair $\mathbf{Q}$, it follows that the expectation value with respect to the Fourier states of the commutators of all pairs of such operators are zero. These states are said to be tracial with respect to this operator algebra, in analogy with the trace property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

The conjugation operator J is the same in the quantum case as in the classical case, which implies that the quantum state functions are real, since they are invariant under the action of J, but not necessarily positive. We will not present an explicit criterion defining exactly the cone of quantum states, a problem similar to that of delimiting the set of Wigner functions representing physical states. But we will make use of the following observation:

Each quantum state function $f(\mathbf{q})$ can be written as the difference between two classical (positive) state functions. Conversely, each classical state function can be written as a difference between two quantum state functions.

This statement can be deduced trivially from the fact that both the quantum and the classical state functions are real, and that the classical state cone contains all positive functions.

3. THE QUANTUM ALGEBRA

To show the connection between the formalism presented above and that of Benatti *et al.* (1991a,b), we will consider the algebra of bounded operators generated by $\mathbf{Q}$. We define the unitary group of unit (periodic) translations $\{W(\mathbf{n})\}$ which generates the relevant algebra:

$$W(\mathbf{n}) = \exp(2\pi i \mathbf{Q}\cdot\mathbf{n}); \qquad \mathbf{n} = (n, m) \in \mathbf{Z}^2 \tag{3.1}$$

It follows from the commutation relation of Q, P that

$$W(\mathbf{n})W(\mathbf{n}') = W(\mathbf{n} + \mathbf{n}')e^{2\pi i\theta(nm' - mn')} \tag{3.2}$$

The algebra of the classical system is generated by the Abelian group $\{W_0(\mathbf{n})\}$:

$$W(\mathbf{n}) = \exp(2\pi i \mathbf{Q}_0\cdot\mathbf{n}); \qquad \mathbf{n} \in \mathbf{Z}^2 \tag{3.3}$$

Using the definition (3.1), it is easily seen that

$$W(\mathbf{n})|\mathbf{k}] = |\mathbf{k} + \mathbf{n}]e^{2\pi i\theta(km - ln)} \tag{3.4}$$

Consequently, the Fourier basis is obtained by applying the operators of this set on the uniform state:

$$W(\mathbf{n})|\mathbf{0}] = |\mathbf{n}] \tag{3.5}$$

Since the Fourier states form a basis for the Hilbert space, it follows that the group $\{W(\mathbf{n})\}$ generates the operator algebra on that space.

4. THE ARNOL'D CAT MAP

The Arnol'd cat map (ACM) dynamics is generated by the operator

$$U|\mathbf{q}] = |\mathbf{T}\mathbf{q}(\text{mod } 1)] \tag{4.1}$$

where $\mathbf{T}$ is the two-by-two matrix $\mathbf{T} = (1, 1; 1, 2)$. The matrix $\mathbf{T}$ is symmetric, isometric, and nonorthogonal. The dynamics $\mathbf{q}(k + 1) = \mathbf{T}\mathbf{q}(k)$ (mod 1) on the phase space is known to be chaotic in every sense of the word. It is chaotic in the topological sense of Devaney (1989), it is ergodic, mixing, and a K-system in the algebraic and entropic sense, and in the sense of

Sagdeev *et al.* (1988). The points with rational coordinates on T^2 constitute a dense subset of periodic points on the torus. A very enlightening and detailed analysis of the behavior of the representative $f(\mathbf{q})$ under the repeated application of U is given by Ford *et al.* (1990).

The action of U on the Fourier basis can be properly described in the bra space,

$$[\mathbf{k}|U = [\mathbf{T}\mathbf{k}| \tag{4.2}$$

This can equivalently be written

$$[\mathbf{T}^{-n}\mathbf{k}|U^n = [\mathbf{k}| \tag{4.3}$$

Moreover, $\mathbf{T}^{-n}\mathbf{k}$ (except for $\mathbf{k} = \mathbf{0}$) has values of increasing absolute value as $n \to \infty$. For a state $|f]$, the Fourier components $[\mathbf{k}|f]$ for high values of $\mathbf{k}$ are increasingly small. However, as n increases, the high values of the low Fourier components are moved upward to the higher components, making the function less smooth. In exchange for this, $[\mathbf{k}|U^n|f]$ for a given $\mathbf{k}$ obtains its value from higher and higher Fourier components of the original function, approaching zero as $n \to \infty$. Eventually, the only component not vanishing in the limit is $[\mathbf{0}|f]$. Thus,

$$\begin{aligned} [g|U^n|f] &= \sum_{\mathbf{k}} [g|\mathbf{k}][\mathbf{k}|U^n|f] \\ &= \sum_{\mathbf{k}} [g|\mathbf{k}][\mathbf{T}^n\mathbf{k}|f] \\ &\to [g|\mathbf{0}][\mathbf{0}|f] = [g|P|f] \end{aligned} \tag{4.4}$$

This is the property of mixing, and can be expressed by the correlator $[f|U^n - P|g]$,

$$[f|U^n - P|g] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.5}$$

Actually, more is known about this limit (Sagdeev *et al.*, 1988, p. 168):

$$[f|U^n - P|g] = Ae^{-\lambda n} \tag{4.6}$$

where $\lambda = \ln[\frac{1}{2}(3 + 5^{1/2})]$ and $A = [f|I - P|g]$. This is an expression of the local exponential separation in $\mathbf{q}$ space with an exponential factor equal to h. This constitutes the K-property of Sagdeev *et al.* (1988).

Note that, since each quantum state can be written as a difference between two classical states, the same type of exponential decay as expressed in (3.6) is valid in the quantum case. Thus, in both cases we have an exponentially fast decay of correlations, reflecting the local exponential separation of the paths of the Arnol'd cat map. The difference between the two cases consists only in the choice of initial states $|f]$, $|g]$ of the correlator. Thus, this strong stochastic property is preserved for all values of h.

5. CONCLUSION

We have seen that the model of Benatti *et al.* is an example of a Tomita representation, a fact which makes it possible to identify elements of the model sustaining a detailed physical interpretation (state vectors, operators of physical quantities). Moreover, we have pointed out that the quantized system still has the strong stochastic property of exponential decay of correlation between the state functions, a property which in the classical case is regarded as an indicator of chaos.

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